

What Graphs are 2-Dot Product Graphs?*

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Abstract. Let $d \geq 1$ be an integer. From a set of d -dimensional vectors, we obtain a d -dot product graph by letting each vector \mathbf{a}^u correspond to a vertex u and by adding an edge between two vertices u and v if and only if their dot product $\mathbf{a}^u \cdot \mathbf{a}^v \geq t$, for some fixed, positive threshold t . Dot product graphs can be used to model social networks. Recognizing a d -dot product graph is known to be NP-hard for all fixed $d \geq 2$. To understand the position of d -dot product graphs in the landscape of graph classes, we consider the case $d = 2$, and investigate how 2-dot product graphs relate to a number of other known graph classes.

1 Introduction

Consider a social network in which each individual is friends with zero or more other individuals. In a vector model of the network, an individual u is described by a d -dimensional vector \mathbf{a}^u for some integer $d \geq 1$ that expresses the extent to which u has each of a set of d attributes (which might, for example, represent their hobbies, political opinions or musical tastes). Then two individuals are assumed to be friends if and only if their attributes are “sufficiently similar”. There are many ways to measure similarity using a vector model (see, for example, [1, 5, 10, 11, 16]). In this paper, we use the *dot product model*: two individuals u and v are friends if and only if the dot product $\mathbf{a}^u \cdot \mathbf{a}^v \geq t$, for some fixed, positive threshold t . The corresponding graph G , in which each individual is a vertex and the friendship relation is described by the edge set, is called a *dot product graph* of *dimension* d or a *d -dot product graph*. We also say that the vector model $\{\mathbf{a}^u \mid u \in V\}$ with the threshold t is a *d -dot product representation* of G .

Dot product graphs have been studied from various perspectives. In particular, the study of dot product graphs as a model for social networks was initiated in a randomized setting [13–15, 17, 18], where the dot product of two vectors gives the probability that an edge occurs between the corresponding vertices. In

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a recent paper [6], we started the study of dot product graphs from an algorithmic perspective by considering the problems of finding a maximum independent set or a maximum clique in a d -dot product graph.

Fiduccia et al. [4] initiated a study of dot product graphs from a graph-theoretic perspective. They showed that every graph on n vertices and m edges has a dot product representation of dimension at most the minimum of $n - 1$ and m . They also introduced the natural notion of the *dot product dimension* of a graph, which is the smallest d such that the graph has a d -dot product representation. Graphs of dot product dimension 1 are easily understood and can be recognized in polynomial time: they are precisely the disjoint union of at most two threshold graphs [4]. This situation changes for higher values of the dot product dimension. Kang and Müller [9] proved that recognizing graphs of any fixed dot product dimension $d \geq 2$ is NP-hard. However, membership in NP is still open for $d \geq 2$ and, in fact, we know comparatively little about graphs of dot product dimension 2 (or any higher fixed value). In particular, we lack an in-depth understanding of how 2-dot product graphs fit within the landscape of known graph classes.

1.1 Previous Work

There are a few previous studies that considered graphs of small dot product dimension. For the definitions of standard graph classes, we refer the reader to [2]. Fiduccia et al. [4] proved that every interval graph and every caterpillar is a 2-dot product graph; note that not all 2-dot product graphs are interval graphs (as the cycle on four vertices is a 2-dot product graph but not an interval graph). Fiduccia et al. [4] also showed that not every tree is a 2-dot product graph, but that trees do have dot product dimension at most 3. Chordal graphs are known to have dot product dimension at most $\min\{\omega(G) + 1, n/2\}$ [4, 12].

Since there exist trees of dot product dimension 3, neither all outerplanar nor all planar graphs are 2-dot product graphs. However, Kang et al. [8] proved tight bounds of 3 and 4, respectively, on the dot product dimension of these graphs; they also showed that every planar graph of girth at least 5 has dot product dimension 3 and that this does not hold if the girth is 4. Li and Chang [12] showed that every wheel on six or more vertices has dot product dimension 3 (a *wheel* is a graph obtained from a cycle by adding a dominating vertex).

Fiduccia et al. [4] proved that every bipartite graph has dot product dimension at most $\frac{n}{2}$ and that this is a tight bound, since the complete bipartite graph $K_{n,n}$ on $2n$ vertices has dot product dimension n . In fact, they conjectured that any graph on n vertices has dot product dimension at most $\frac{n}{2}$. Li and Chang [12] confirmed this conjecture for chordal graphs, graphs with girth at least 5, and P_4 -sparse graphs.

Observe that, in spite of these results, there are still many graph classes for which the relation to the class of graphs of small dot product dimension (and of dot product dimension 2 in particular) is unclear.

1.2 Our Results

We provide a more complete picture of the place of 2-dot product graphs in the landscape of known graph classes. To this end, we identify several new graph classes that are 2-dot product graphs. We also show that certain graph classes are neither contained in the class of 2-dot product graphs nor do they contain all 2-dot product graphs. In particular, our work provides evidence that no well-known graph class includes all 2-dot product graphs (however, we note explicitly here that we neither claim nor conjecture this).

In Section 2.1 we state several observations on 2-dot product graphs. Then, in Section 2.2, we consider *co-bipartite graphs* (complements of bipartite graphs): a complete graph minus a matching remains a 2-dot product graph, but we show that there exist co-bipartite graphs with dot product dimension greater than 2. An *intersection graph* of any collection of sets has the sets as its vertex set and its edges representing when pairs of sets intersect. A *unit circular-arc graph* is the intersection graph of unit-length arcs of a circle. A *unit disk graph* is the intersection graph of unit-size disks in the plane. In Section 2.3, we prove that not all unit circular-arc graphs have dot product dimension 2, but give sufficient conditions for them to have dot product dimension 2. In the same section, we also show that unit disk graphs may not have dot product dimension 2, but do have dot product dimension at most 3. Note that the latter bound is sharp, as the 6-wheel, which has dot product dimension 3 [12], is a unit disk graph (see [3]). Finally, in Section 2.4, we consider *split graphs*; that is, graphs whose vertices can be partitioned into two sets that induce an independent set and a clique. We show the existence of split graphs with dot product dimension greater than 2.

2 Graph Classes and 2-Dot Product Graphs

2.1 Observations

Throughout, we assume that the threshold $t = 1$, unless stated otherwise. As any d -dot product graph has a representation with threshold 1 [4], this is no restriction.

We start by observing that because the class of 2-dot product graphs is closed under vertex deletion, it can be characterized by a set of forbidden induced subgraphs. However, the class of 2-dot product graphs is not well-quasi-ordered, that is, it has no *finite* set of forbidden induced subgraphs, because every wheel must be in this set of forbidden induced subgraphs. Indeed, a wheel minus a vertex is either a cycle or a fan, and thus has dot product dimension 2.

We note that 2-dot product graphs are not necessarily triangle-free, planar, nor H -minor-free for some fixed H , as they can contain arbitrarily large cliques. They are also not necessarily split, AT-free, even hole-free, or odd hole-free, because cycles of any length are 2-dot product (see [4] for cycles of length 4 or length at least 6; for the 5-vertex cycle this follows from Li and Chang’s result [12] on graphs with girth at least 5). Also 2-dot product graphs are neither necessarily

claw-free, as the claw has a 2-dot product representation (for example, take $t = 3$ and vectors $(1, 1)$, $(1, 1)$, $(1, 1)$ and $(2, 2)$), nor circular-arc (for example, take the complete graph on four vertices and add a pendant vertex to each vertex). Moreover, there exist 2-dot product graphs that are not a disk graph (take the bi-4-wheel which can be represented as $(0, 5)$, $(\frac{1}{5}, 2)$, $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2})$, $(2, \frac{1}{5})$, $(5, 0)$ with $t = 1$). We note that grid graphs are not 2-dot product, as the 2×2 grid can easily be shown not to have a 2-dot product representation by following the proof for wheels in [12]. Because C_5 has a 2-dot product representation, we know that 2-dot product graphs are not necessarily perfect graphs, nor even circular perfect.

We need some definitions and four lemmas, some of which are known already. We say that a vertex v is *between* vertices u and w if \mathbf{a}^v can be written as a nonnegative linear combination of \mathbf{a}^u and \mathbf{a}^w . In other words, v is between u and w if \mathbf{a}^v lies in the plane defined by \mathbf{a}^u and \mathbf{a}^w and \mathbf{a}^v lies within the smaller of the two angles defined by \mathbf{a}^u and \mathbf{a}^w in this plane.

Lemma 1 ([4]). *If a, b, c and d are vertices in a graph with a 2-dot product representation and ad and bc are edges but ac and bd are not, then b and c are not both between a and d .*

Lemma 2 ([6]). *If a, b and c are vertices in a graph with a 2-dot product representation and c is between a and b and ab is an edge but ac is not, then $|\mathbf{a}^b| > |\mathbf{a}^c|$.*

Lemma 3. *If a, b, c and d are vertices in a graph with a 2-dot product representation and ab and cd are edges but ac and bd are not, then if c is between a and b , then b is not between c and d .*

Proof. By Lemma 2 we have that if c is between a and b , then $|\mathbf{a}^b| > |\mathbf{a}^c|$. But, if b is between c and d , then, using the lemma again considering b, c and d , we have $|\mathbf{a}^c| > |\mathbf{a}^b|$. \square

Lemma 4. *If a, b, c and d are vertices in a graph with a 2-dot product representation and induce a 4-cycle with edges ab, bc, cd and ad , then $\mathbf{a}^a, \mathbf{a}^b, \mathbf{a}^c$ and \mathbf{a}^d are in a half-plane. Moreover in the linear ordering given by the size of the angles from one of the bounding half-lines, the first and fourth vertices, and the second and third vertices, are non-adjacent.*

Proof. By Lemma 1, b and c are not both between a and d . So we can assume, without loss of generality, that b is not between a and d . Thus either d is between a and b or a is between b and d .

If d is between a and b , then, by Lemma 1, c is not between a and b , and by Lemma 3, a is not between c and d . So b and d are between a and c and we have the required ordering. Noting also that b and d are first and fourth in the ordering and have common neighbours implies that the four vectors lie in a half-plane.

Finally if a is between b and d , then, by Lemma 1, a and b are not both between c and d and a and d are not both between b and c . So a and c are both between b and d and again we have the required ordering. \square

2.2 Co-Bipartite Graphs

We exhibit a sharp divide on whether co-bipartite graphs are 2-dot product graphs. First, we show that a complete graph minus a matching is still a 2-dot product graph.

Theorem 1. *Let G be a graph obtained from a complete graph by removing the edges of a matching. Then G has a 2-dot product representation.*

Proof. Let m be a positive integer. Let the vertex set of K_{2m} be denoted $\{v_1, v_2, \dots, v_m, w_1, w_2, \dots, w_m\}$ and let I_m denote the set of edges $\{v_i w_i \mid 1 \leq i \leq m\}$ which form a perfect matching. We will first prove the special case of the theorem where $G = K_{2m} - I_m$. For a nonnegative integer k , let $b(k) = 2^k - 1$. For $1 \leq i \leq m$, let $\mathbf{a}^{v_i} = (1/b(i), b(i-1))$, $\mathbf{a}^{w_i} = (b(i-1), 1/b(i))$. We show this is a 2-dot product representation for $K_{2m} - I_m$. First consider pairs v_i, w_i :

$$\mathbf{a}^{v_i} \cdot \mathbf{a}^{w_i} = \frac{2b(i-1)}{b(i)} = \frac{2^i - 2}{2^i - 1} < 1.$$

We must show that all other pairs of distinct vertices have dot product at least 1. As each $b(k) \geq 1$, we have $\mathbf{a}^{v_i} \cdot \mathbf{a}^{v_j} \geq 1$ and $\mathbf{a}^{w_i} \cdot \mathbf{a}^{w_j} \geq 1$ for all i, j . Finally, for $i \neq j$,

$$\mathbf{a}^{v_i} \cdot \mathbf{a}^{w_j} = \frac{b(j-1)}{b(i)} + \frac{b(i-1)}{b(j)},$$

and one of the two quotients is at least 1, and the other is positive.

For the general case, choose the largest value of m such that $K_{2m} - I_m$ is an induced subgraph of G . Then every vertex not in this subgraph is adjacent to every vertex other than itself. We can obtain a 2-dot product representation of G using the representation described above for the vertices of $K_{2m} - I_m$ and by letting, for every other vertex u , $\mathbf{a}^u = (1, 1)$ (and by noting that every vertex has two positive coordinates one of which is at least 1). \square

We show that, in contrast to Theorem 1, almost all even anti-cycles have dot product dimension more than 2. Since even anti-cycles are co-bipartite, in any 2-dot product representation of an even anti-cycle, there is an empty quadrant and so we can assume a linear ordering \prec on the vertices. For a positive integer $n \geq 3$, let A_{2n} be the (even) anti-cycle with vertices $v_1, \dots, v_n, w_1, \dots, w_n$ where $v_1 \prec v_2 \dots v_n \prec w_n \prec w_{n-1} \dots \prec w_1$. We say that the ordering \prec is *nested* if

- (N1) $v_1 w_1, v_1 w_2$ and $v_2 w_1$ are non-edges;
- (N2) for $2 \leq i \leq n-1$, $v_i w_{i-1}, v_i w_{i+1}, w_i v_{i-1}$ and $w_i v_{i+1}$ are non-edges.
- (N3) $v_{n-1} w_n, v_n w_n$ and $v_n w_{n-1}$ are non-edges.

The three conditions describe the edge set of A_{2n} in terms of the ordering \prec . In fact, there is redundancy as most of the edges are described twice. Notice that (N1) is concerned with the adjacencies of the vertices at each end of the ordering, (N3) tells us about the middle two vertices and (N2) gives the non-neighbours of the vertices at distance i from each end of the ordering (for $2 \leq i \leq n-1$).

Lemma 5. *For $n \geq 3$, if A_{2n} has a 2-dot product representation, then it is nested.*

Proof. We suppose that we have a 2-dot product representation for A_{2n} and label the vertices so that $v_1 \prec v_2 \dots w_1$. We must show that the adjacencies cannot contradict (N1), (N2) and (N3).

We note that (N3) is implied by (N1) and (N2) by considering the degrees of v_n and w_n .

Consider (N1). Suppose that v_1 is adjacent to w_1 . Let a and b be the two non-neighbours of w_1 . As there is no 3-cycle in the complement, ab is an edge, and v_1 cannot be adjacent to both a and b else there is a 4-cycle in the complement. Thus v_1, a, b and w_1 contradict Lemma 1.

Suppose that v_1 is adjacent to w_2 . Let a be the non-neighbour of v_1 that is not w_1 and so $a \prec w_2$. Suppose that $w_2 w_1$ is not an edge. Then, by Lemma 3, aw_1 is not an edge, but now w_1 has three non-neighbours — v_1, a and w_2 . So $w_2 w_1$ is an edge. Now we can, essentially, repeat the argument of the previous paragraph: let c and d be the non-neighbours of w_2 and v_1, c, d, w_2 contradict Lemma 1,

So the non-neighbours of v_1 are w_1 and w_2 , and, by symmetry, the non-neighbours of w_1 are v_1 and v_2 .

We now prove (N2) is true for each i by induction. Note that we already have that $v_i w_{i-1}$ and $v_{i-1} w_i$ are not edges by considering (N2) for $i-1$ or, in the case $i=2$, using (N1). We must show that $v_i w_{i+1}$ is not an edge (and then, by symmetry, we will be done).

Consider the set of vertices $V_i = \{v_1, \dots, v_i, w_1, \dots, w_i\}$. Each vertex in $V_i \setminus \{v_i, w_i\}$ has two non-neighbours in V_i . Suppose that $v_i w_{i+1}$ is an edge. If $w_{i+1} w_i$ is also an edge, then each vertex in $V_i \cup w_{i+1}$ has two non-neighbours in the set and so in the complement these vertices induce a set of cycles; a contradiction (in fact, it is easy but not necessary to check that they induce a single cycle).

So let a and b be the non-neighbours of w_{i+1} , and note that they must be between v_i and w_{i+1} . Again a and b are adjacent to avoid a 3-cycle in the complement and v_i must be adjacent to at least one of them as we know one of its non-neighbours is w_{i-1} . So v_i, a, b and w_{i+1} contradict Lemma 1 and we are done. \square

Theorem 2. *For $n \leq 3$, A_{2n} is a 2-dot product graph. For $n \geq 4$, A_{2n} is not a 2-dot product graph.*

Observe that A_6 is 2-dot product and as such does have a nested representation, for example:

$v_1: (5, 0), v_2: (3, 1/6), v_3: (1/2, 1/4), w_3: (1/4, 1/2), w_2: (1/6, 3), w_1: (0, 5)$

By Lemma 5, this is the only possible representation (up to isomorphism).

For $n \geq 4$, A_{2n} does not have a nested representation — consider the vertices v_2, v_3, w_3, w_2 and apply Lemma 1 — and so does not have any 2-dot product representation. \square

Corollary 1. *There exist co-bipartite graphs that do not have a 2-dot product representation.*

2.3 Unit Circular-Arc Graphs

Consider the unit sphere S^k . Then for some vector $\mathbf{c} \in S^k$, a *cap* of S^k is the set $\{\mathbf{x} \in S^k \mid \mathbf{c} \cdot \mathbf{x} \geq a\}$, where a is a real number in $(0, 1]$. We call the vector \mathbf{c} the *centre* of the cap, and $2 \arccos a$ its *angular diameter*. Observe that, given the range of a , the angular diameter of each cap lies in $[0, \pi)$. Fiduccia et al. [4] considered the capture graph of caps of S^k : a graph G is a capture graph if one can assign to each vertex a cap on S^k so that if a pair of vertices are adjacent the centre of one cap is contained in the other and if they are not adjacent the caps are disjoint. They showed that such a graph has dot product dimension at most $k + 1$. Kang et al. [8] studied the contact graph of caps: a graph G is a contact graph if one can assign to each vertex a cap on S^k so that if a pair of vertices are adjacent the caps intersect in a single point and if they are not adjacent the caps are disjoint. They showed that such a graph has dot product dimension at most $k + 2$. We consider *unit caps*: a set of caps of S^k is unit if all caps in the set have the same angular diameter $\theta \in [0, \pi/2)$.

Theorem 3. *The intersection graph of a set of unit caps of S^k has dot product dimension at most $k + 1$.*

Proof. Let $\mathcal{C} = \{C_1, \dots, C_n\}$ denote a set of unit caps of S^k , let \mathbf{c}_i denote the centre vector of C_i , and let $\theta \in [0, \pi/2)$ denote the common angular diameter of the caps. Define $\mathbf{a}_i = \frac{1}{\sqrt{\cos \theta}} \mathbf{c}_i$ and let $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$. Since $\theta \in [0, \pi/2)$, \mathcal{A} is properly defined.

Now observe that if C_i and C_j intersect, then the angle between \mathbf{c}_i and \mathbf{c}_j is at most θ . Hence,

$$\mathbf{a}_i \cdot \mathbf{a}_j \geq \left(\frac{1}{\sqrt{\cos \theta}} \right)^2 \cos \theta = 1.$$

If C_i and C_j do not intersect, then the angle between \mathbf{c}_i and \mathbf{c}_j is larger than θ . Hence,

$$\mathbf{a}_i \cdot \mathbf{a}_j < \left(\frac{1}{\sqrt{\cos \theta}} \right)^2 \cos \theta = 1.$$

It follows that the intersection graph G of \mathcal{C} is isomorphic to the dot product graph of \mathcal{A} . Since the vectors in \mathcal{A} lie in \mathbb{R}^{k+1} , the dot product dimension of G is at most $k + 1$. \square

Corollary 2. *Unit disk graphs have dot product dimension at most 3.*

Proof. By scaling and a stereographic projection onto S^2 , it can be seen that each unit disk graph is the intersection graph of unit caps of S^2 . The result follows from Theorem 3. \square

It would seem that Theorem 3 also implies that all unit circular-arc graphs have dot product dimension at most 2. However, due to the limited angular diameter allowed in our definition of unit caps, this implication only holds if the graph has a unit circular-arc representation using unit caps of S^1 . This is the case, for example, when the graph has no maximal independent set of size less than 4.

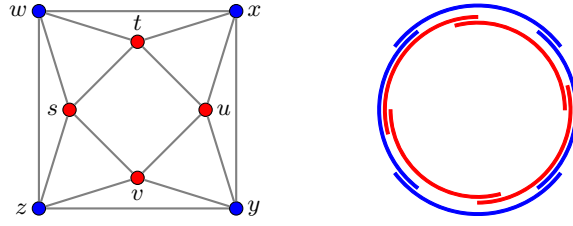


Fig. 1. The graph J and its representation as a unit circular-arc graph.

Theorem 4. *If G is a unit circular-arc graph with no maximal independent set of size less than 4, then G is a 2-dot product graph.*

Proof. A cap of S^1 is essentially equal to an arc of the circle. Our definition of unit caps limits the angular diameter of unit caps to at most $\frac{1}{2}\pi$, that is, unit arcs that each cover at most $\frac{1}{4}$ of the circle. So in order for the proof to work, we need to ensure that the unit arcs in a representation of the unit circular-arc graph each cover at most $\frac{1}{4}$ of the circle. This is guaranteed to be the case, for example, when the graph has no maximal independent set of size less than 4 (if the graph has a maximal independent set of size 4 or more, then necessarily each arc covers less than $\frac{1}{4}$ of the circle. \square)

Surprisingly, the restriction on the size of a maximal independent set in Theorem 4 is not an artifact of our proof technique, but is actually needed: in Figure 1 is an example of a graph J that is a unit circular-arc graph and that has dot product dimension larger than 2 (this will be shown in the proof of Theorem 5). Note that such an example must have triangles, due to the following proposition.

Proposition 1. *Any triangle-free unit circular-arc graph is isomorphic to a path or a cycle. Therefore it has dot product dimension 2.*

Proof. Suppose that G is a triangle-free unit circular-arc graph and is not isomorphic to a path. Since unit circular-arc graphs are claw-free, G is not a tree and so contains a cycle. Let C be a shortest induced cycle of G . Since G is triangle-free, $|C| \geq 4$. Then the arcs of C must cover the circle. Since G is claw-free, any arc not on the cycle must intersect at least two arcs of the cycle, creating a triangle. Hence, G is isomorphic to C . \square

Theorem 5. *There exist unit circular-arc graphs that do not have a 2-dot product representation.*

Proof. It is sufficient to show that the graph J of Figure 1 does not have a 2-dot product representation. Suppose a representation exists. By Lemma 4, we can assume that $\mathbf{a}^s, \mathbf{a}^t, \mathbf{a}^u$ and \mathbf{a}^v are in a half-plane and that if \prec is the linear ordering given by the size of the angles from one of the bounding half-lines, then

$s \prec t \prec v \prec u$. We can include w, x, y and z in this ordering: if a vertex a is not in the half-plane, then we say it precedes s in the ordering only if \mathbf{a}^a is in the quadrant that precedes s .

We consider the ordering of t, v, x and z and prove the theorem by showing that all possible orderings give a contradiction. By Lemma 1, $t \prec v \prec z \prec x$, is not possible. By Lemma 3, $t \prec v \prec x \prec z$, is not possible. Thus v cannot precede both x and z , and, by the symmetry of J , x and z cannot both precede t .

Suppose that neither of x and z is between t and v . As, by Lemma 3, $z \prec t \prec v \prec x$ is impossible, we must have $x \prec t \prec v \prec z$. But if $s \prec x$, then s, x, t and z contradict Lemma 1; if $x \prec s$, then x, s, v, u provide the contradiction.

So one of x and z must be between t and v . Without loss of generality we can assume $t \prec x \prec v$. If $x \prec z$, then s, t, x and z contradict Lemma 1. If $t \prec z \prec x \prec v$, then Lemma 3 is contradicted. Finally, if $z \prec t$, then z, t, x and v provide the contradiction. \square

In line with Corollary 2, we note that J does indeed have a 3-dot product representation: $(2, 0, 1), (0, 2, 1), (-2, 0, 1), (0, -2, 1), (1, 1, -1), (1, -1, -1), (-1, -1, -1), (-1, 1, -1)$.

2.4 Split Graphs

We prove the following result.

Theorem 6. *There exist split graphs that do not have a 2-dot product representation.*

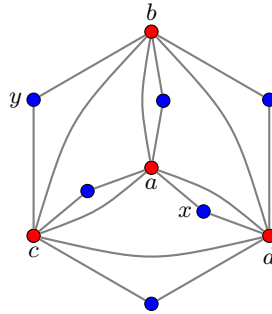


Fig. 2. The graph K . As the vertices can be partitioned into a clique and an independent set, K is a split graph.

Proof. It is sufficient to show that the graph K of Figure 2 does not have a 2-dot product representation. The vector representations of the vertices a, b, c and d

that form a clique must be contained in a quadrant and as every other vertex is adjacent to the clique the vector representations are contained within three quadrants. So there is some vertex such that there is no other vector within $\pi/2$ moving anticlockwise. Let \prec be a linear ordering of the vertices according to their clockwise angle from this “first” vertex.

Assume that $a \prec b \prec c \prec d$. Suppose that $d \prec x$. Then $y \prec x$ (by Lemma 1 considering b, d, x and y). But if $a \prec y \prec d$ then these three vertices and x together contradict Lemma 1, and if $y \prec a$, then y, a, b and x contradict Lemma 4.

So $x \prec d$, and, by symmetry, $a \prec x$. Suppose that $c \prec x \prec d$. If $d \prec y$, then b, x, d and y contradict Lemma 1, and if $y \prec x$, then a, b, y and x provide a similar contradiction. So we must have $x \prec y \prec d$, but now b, x, y and d and Lemma 4 provide a contradiction.

Thus we cannot have x between c and d which also shows that x cannot be between a and b . The only possibility that remains is that $b \prec x \prec c$. If $y \prec a$, then y, a, b and x contradict Lemma 4. And if $a \prec y \prec x$, then these three vertices and b contradict Lemma 1. Finally, if $x \prec y$, we note that by symmetry we can already assume that $y \prec d$, and we use Lemma 1 with b, x, y and d to show that the representation cannot exist. \square

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